

HÖJDPUNKTEN 2023

Gymnasietävling, 10-11 of March 2023



Time: 3 hours

Tools: Only pen, eraser, compass and ruler

Full written solutions are required, in general no points are given for only writing the answer.

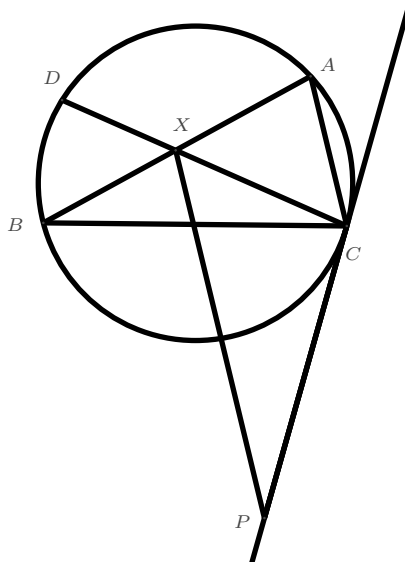
Problem 1. There are 100 people sitting in a circle. We know that some of them are liars who always lie, and some of them are truth tellers who always tell the truth, but we do not know who is what. Everyone in the circle says that they are sitting next to at least one liar. What is the smallest and largest possible number of liars in the circle?

Problem 2. Determine all integers n for which $n^4 - 3n^2 + 9$ is a prime number.

Problem 3. Determine all pairs of positive integers (m, n) such that it's possible to build a rectangle using m vertical dominoes and n horizontal dominoes. A domino is made up of two squares of the same size, glued together along one of their edges. You may not rotate the tiles and they are all the same size.

Problem 4. The students in the UVS-village live in n skyscrapers which are evenly spaced along the main street. One student lives in the first house, two students live in the second house, and so on, until house n where n students live. When it's time to organise a big maths competition, the organisers want to know where to host it in order to minimise the total travel distance for all the students. Which house should they choose? Note that the answer may depend on n .

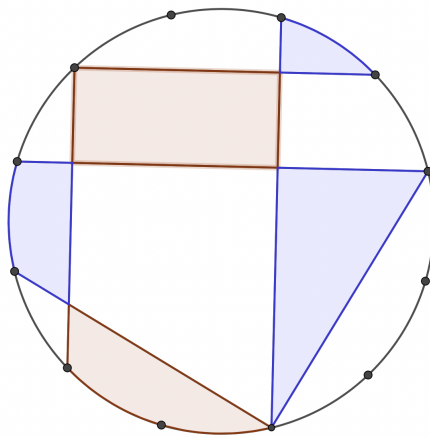
Problem 5. Let ABC be a triangle, and let (ABC) be its circumcircle. Let P be a point outside the circle such that PC is tangent to (ABC) , and $|PC| = |BC|$. Let X be a point on the chord AB such that PX is parallel to AC , and let the line CX intersect (ABC) in the point D . Show that $|BD| = |DX|$.



Problem 6. Let $n \leq 100$ be a positive integer. We calculate the numbers $0 \cdot n, 1 \cdot n, \dots, 100 \cdot n$ in this order, and write down their remainders when dividing by 101 in a row. For how many pairs of adjacent numbers in the row is the left one bigger than the right one?

Problem 7. The numbers $1, 2, 3, \dots, 1000$ are written on the board. Kevin picks 12 of them, and erases the rest. He then notes that no sum of some numbers left on the board is a perfect power. Is this possible?

Problem 8. Prove that the sum of the blue areas is equal to the sum of the red areas. The figure is a circle and the points are evenly spaced.



Problem 9. Sara splits the numbers $404, 405, \dots, n$ into two groups. Each number must be in exactly one group, and the groups are allowed to have different size. Then Anton tries to find three numbers x, y, z that are in the *same* group such that $x + y = z$. What is the smallest positive integer n such that Anton can do this, regardless of how Sara split the numbers into two groups?

Problem 10. Let k be a positive integer, and let S be a set containing $2^k + 1$ different positive integers. We say that a prime is *nice* if it divides the sum of two distinct numbers in S . Show that there are at least $k + 1$ *nice* primes.

Example: If $k = 1$ and $S = \{1, 3, 7\}$, then $1 + 3 = 4$, $1 + 7 = 8$ and $3 + 7 = 10$ are all possible sums of two distinct numbers in S . Hence 2 and 5 are nice primes (since 2 divides 4 and 5 divides 10). In this case there are no other nice primes.

Problem 11. In the land far, far away two teams are competing - the red team and the blue team. There are n cities, some pairs of which are connected by roads. In the beginning, the roads don't belong to anyone. The teams then take turns picking a road that has not yet been picked, and colour it with their own colour. The red team makes the first pick.

If at any point it's possible to travel between any pair of cities only using blue roads, the blue team wins (note that it's allowed to use multiple roads to travel between two cities, as long as they are all blue). If all the roads have been picked (by some team) without the blue team achieving this, the red team wins.

Prove that the blue team can guarantee a win if and only if it's possible to split the roads into two groups, such that within each group it's possible to travel between any pair of cities.